

ALGORITHMS FOR EMBEDDED MONOIDS AND BASE POINT FREE PROBLEMS

ANNE FAHRNER

ABSTRACT. We present algorithms for basic computations with monoids in finitely generated abelian groups such as monoid membership testing and computing an element of the conductor ideal. Applying them to Mori dream spaces, we obtain algorithms to test whether a Weil divisor class of a given Mori dream space is base point free, to compute generators of the monoid of base point free Cartier divisor classes and to test whether a Mori dream space with known canonical class fulfills Fujita's base point free conjecture or not.

1. INTRODUCTION

A first part of this paper concerns *embedded monoids*, that means finitely generated monoids in finitely generated abelian groups, and thereby generalises ideas of the theory on affine semigroups [5, Chapter 2] to monoids with non-trivial torsion part. We further present algorithms for embedded monoids, among others for computing generators of intersections of embedded monoids and for computing an element of the conductor ideal; see Algorithms 2.4 – 2.8.

In the second part of the paper, we apply these algorithms to base point free questions for Mori dream spaces. Recall that Mori dream spaces, introduced by Hu and Keel [12], are characterized via their optimal behaviour with respect to the minimal model program. A particular interesting aspect of Mori dream spaces is their highly combinatorial structure [1] – in this regard they are a canonical generalisation of toric varieties. Further well known example classes are spherical varieties [4], smooth Fano varieties [2] and all Calabi-Yau varieties with polyhedral effective cone [15]. The combinatorial framework developed in [1] allows algorithmic treatment of Mori dream spaces. Applying the aforementioned algorithms to Mori dream spaces, we provide algorithms for testing whether a given Weil divisor class is base point free and for computing generators of the *base point free monoid*, i.e. the monoid of base point free Weil divisor classes; see Algorithms 3.3 and 3.4.

These algorithms, together with the non-emptiness of the conductor ideal of the base point free monoid, see Corollary 3.2, play an important role in our main algorithm, Algorithm 4.5, testing Fujita's base point free conjecture [10]: this much studied conjecture claims that for a smooth projective variety with canonical class \mathcal{K}_X , the Weil divisor class $\mathcal{K}_X + m\mathcal{L}$ is base point free for all ample Cartier divisor classes \mathcal{L} and for all $m \geq \dim(X) + 1$. So far it is known to hold for smooth projective varieties up to dimension five [16, 6, 13, 17]. Despite this substantial progress, Fujita's base point free conjecture remains in general still open. With Algorithm 4.5, we provide a tool for its algorithmic testing for Mori dream spaces. Since our algorithm makes use of the canonical class \mathcal{K}_X , it applies to Mori dream spaces with known \mathcal{K}_X . This case appears quite often: for instance if X is spherical or if its Cox ring is a complete intersection, see Remark 4.2 for details.

2010 *Mathematics Subject Classification.* 14Q15, 20M14.
Supported by the Carl-Zeiss-Stiftung.

In [8], we provide an implementation of our algorithms building on the two Maple-based software packages **convex** [7] and **MDSpackage** [11]. Using this implementation, we proof Fujita's base point free conjecture for a six-dimensional Mori dream space, see Example 4.6. In addition, we study the more general question of the existence of semiample Weil divisor classes that are not base point free. It is well known that for complete toric varieties, semiampleness implies base point freeness. For smooth rational projective varieties with a torus action of complexity one and Picard number two, the same statement follows immediately from the classification done in [9]. In Example 3.5, we present a first example of a smooth surface of Picard number twelve having a semiample Cartier divisor with base points.

The author would like to thank Jürgen Hausen for valuable discussions and comments.

2. EMBEDDED MONOIDS

Let K be a finitely generated abelian group. We denote by $K = K^0 \oplus K^{\text{tor}}$ the decomposition of K into free and torsion part and we write $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$ for the associated rational vector space. Note that each $w \in K = K^0 \oplus K^{\text{tor}}$ can be represented as $w = (w^0, w^{\text{tor}})$ with unique elements $w^0 \in K^0$ and $w^{\text{tor}} \in K^{\text{tor}}$. Every $w \in K$ defines an element $w \otimes 1 \in K_{\mathbb{Q}}$, which we denote as well by w for short. A *cone* in a rational vector space always refers to a convex, polyhedral cone. The relative interior of a cone $\tau \subseteq K_{\mathbb{Q}}$ is denoted by τ° .

By an *embedded monoid* we mean a pair $S \subseteq K$, where S is a finitely generated submonoid of K . For an embedded monoid $S \subseteq K$, we denote by

$$\text{cone}(S) := \text{cone}(w \otimes 1; w \in S) \subseteq K_{\mathbb{Q}}$$

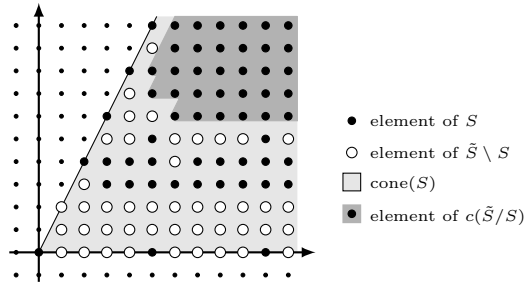
the (convex, polyhedral) cone generated by the elements of S . An embedded monoid $S \subseteq K$ is *spanning*, if S generates K as a group. The *saturation* of an embedded monoid $S \subseteq K$ is the embedded monoid

$$\tilde{S} := \{w \in K; nw \in S \text{ for some } n \in \mathbb{Z}_{\geq 1}\} \subseteq K.$$

Let $S \subseteq K$ be an embedded monoid. A non-empty set $M \subseteq K$ is called an *S-module* if $S + M \subseteq M$ holds. We call an S -module M *ideal*, if $M \subseteq S$ holds and *finitely generated*, if there is a finite subset $\{m_1, \dots, m_{\ell}\} \subseteq M$ with the property that $M = \{m_1 + s, \dots, m_{\ell} + s; s \in S\}$ holds.

Definition 2.1. Let $S \subseteq K$ be an embedded monoid. The *conductor ideal* of $S \subseteq K$ is the set

$$c(\tilde{S}/S) := \{x \in S; x + \tilde{S} \subseteq S\}.$$



Proposition 2.2. *Let $S \subseteq K$ be an embedded monoid. If $S \subseteq K$ is spanning, then the conductor ideal $c(\tilde{S}/S)$ is non-empty, i.e. it is in particular an S -module.*

Proof. By definition, $S + c(\tilde{S}/S) \subseteq c(\tilde{S}/S)$ holds, i.e. we only have to show that $c(\tilde{S}/S)$ is non-empty. In case of a torsion free group K , one can find a proof in [5, Proposition 2.33]. In case of general finitely generated abelian groups, the proof follows similarly. \square

Proposition 2.3. *Let K_1, K_2 be subgroups of a finitely generated abelian group K and consider embedded monoids $S_i \subseteq K_i, i = 1, 2$. If $\text{cone}(S_1)^\circ \cap \text{cone}(S_2)^\circ \neq \emptyset$ holds and $S_i \subseteq K_i$ is spanning for $i = 1, 2$, then $S_1 \cap S_2 \subseteq K_1 \cap K_2$ is a spanning embedded monoid.*

Proof. We denote by S_{12} the intersection of S_1 and S_2 . Clearly, the group generated by S_{12} is contained in $K_1 \cap K_2$. It remains to show the opposite inclusion. Because of $\tau := \text{cone}(S_1)^\circ \cap \text{cone}(S_2)^\circ \neq \emptyset$, we obtain that the rank of $K_1 \cap K_2$ and the dimension of τ coincide. Thus one can show that

$$\text{lin}_{\mathbb{Z}}(\tilde{S}_1 \cap \tilde{S}_2) = K_1 \cap K_2$$

holds. Let $x \in K_1 \cap K_2$ such that $x \otimes 1$ is an element of τ . Since $S_i \subseteq K_i$ is spanning for $i = 1, 2$, some multiple $mx, m \in \mathbb{Z}_{\geq 1}$, is contained in $C := c(\tilde{S}_1/S_1) \cap c(\tilde{S}_2/S_2)$. This means that $mx + (\tilde{S}_1 \cap \tilde{S}_2)$ is contained in C and thus $\text{lin}_{\mathbb{Z}}(C) = K_1 \cap K_2$ holds. Note that C is contained in $c(\widetilde{S_{12}}/S_{12})$. We thus obtain

$$K_1 \cap K_2 = \text{lin}_{\mathbb{Z}}(C) \subseteq \text{lin}_{\mathbb{Z}}(c(\widetilde{S_{12}}/S_{12})) = \text{lin}_{\mathbb{Z}}(S_{12}).$$

\square

In the following we describe some algorithms for monoids which, applied to Mori dream spaces, can be used for computing the base point free monoid $\text{BPF}(X)$, for testing whether a Cartier divisor class is base point free and for computing a point of the conductor ideal of $\text{BPF}(X) \subseteq \text{Pic}(X)$.

Algorithm 2.4 (inMonoid). *Input:* An embedded monoid $S = \text{lin}_{\mathbb{Z}_{\geq 0}}(s_1, \dots, s_t) \subseteq K$ and an element $w \in K$.

- Compute a representation $K \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^s \mathbb{Z}/a_i\mathbb{Z}$.
- By e_i we denote the canonical base vectors of \mathbb{Z}^t . Define the homomorphism $Q: \mathbb{Z}^t \rightarrow K, e_i \mapsto s_i$ and let Q^0 be the free part of Q , i.e. with the projection $\pi: K \rightarrow K^0 = K/K^{\text{tor}}$, we have $\pi \circ Q = Q^0$.
- Compute the polyhedron $\mathcal{B} := (Q^0)^{-1}(w^0) \cap \mathbb{Q}_{\geq 0}^t$.
- If \mathcal{B} is not bounded, then
 - for all $1 \leq j \leq t$ do
 - * if $s_j^0 = 0_{K^0}$ holds, then let $\mathcal{C} := \{1 \leq i \leq s; (s_j)_{r+i} \neq 0\}$ and

$$\mathcal{B} := \mathcal{B} \cap \{x \in \mathbb{Q}^t; x_j \leq \prod_{i \in \mathcal{C}} a_i\}.$$

- Compute the lattice points of the polytope \mathcal{B} , i.e. compute $B := \mathcal{B} \cap \mathbb{Z}^t$.
- Return *true*, if there is a point $x \in B$ such that $Q(x) = w$ holds. Otherwise, return *false*.

Output: *True* if w is contained in S . Otherwise, *false* is returned.

Proof. We need to show that $w \in S$ holds if and only if the algorithm returns *true*. Clearly, $w \in S$ holds if and only if there is $x \in \mathcal{A} := (Q^0)^{-1}(w^0) \cap \mathbb{Z}_{\geq 0}^t$ such that $Q(x) = w$ holds. If \mathcal{A} is a finite set, there is nothing to show. Note that we have

$$\mathcal{A} \subseteq \left\{ x \in \mathbb{Q}^t; x_j \leq \min \left(\left\lfloor \frac{w_k}{(s_i)_k} \right\rfloor; 1 \leq k \leq r, (s_i)_k \neq 0 \right) \right\}$$

for all $1 \leq j \leq t$ such that $s_j^0 \neq 0_{K^0}$ holds. So if \mathcal{A} is unbounded, there is an integer $1 \leq j_0 \leq t$ such that $s_{j_0}^0 = 0_{K^0}$ holds. After renumbering the generators s_1, \dots, s_t of S , we may assume that $s_j^0 = 0_{K^0}$ holds if and only if j is at least j_0 . It follows that for some polytope $P \subseteq \mathbb{Q}^t$, we have $\mathcal{A} \subseteq P \times \text{lin}(e_j; j_0 \leq j \leq t)$. Let $j_0 \leq j \leq t$ and consider $\mathcal{C} := \{1 \leq i \leq s; (s_j)_{r+i} \neq 0\}$ as well as

$$b_j := \prod_{i \in \mathcal{C}} a_i.$$

Then $\alpha s_j \equiv (\alpha \bmod b_j) s_j$ holds for all $\alpha \in \mathbb{Z}$. Hence there is a point $x \in \mathcal{A}$ such that $Q(x) = w$ holds if and only if there is a point $x \in \mathcal{A} \cap \{x \in \mathbb{Q}^t; x_j \leq b_j\}$ such that $Q(x) = w$ holds. This completes the proof. \square

Algorithm 2.5 (generatorsIntMonoid). *Input:* A finitely generated abelian group K and embedded monoids $S_i := \text{lin}_{\mathbb{Z}_{\geq 0}}(s_{i1}, \dots, s_{in_i}) \subseteq K_i$, $i = 1, 2$, where $K_1, K_2 \subseteq K$ are subgroups.

- Let $\varphi := \varphi_1 \times \varphi_2: \mathbb{Z}^{n_1+n_2} \rightarrow K \times K$ be the homomorphism of abelian groups defined through $\varphi_i: \mathbb{Z}^{n_i} \rightarrow K, e_{ij} \mapsto s_{ij}$, where the e_{ij} denote the canonical base vectors of \mathbb{Z}^{n_i} . Furthermore, define the projection $\psi: K \times K \rightarrow (K \times K)/\Delta$, where $\Delta := \{(k, k); k \in K\}$ denotes the diagonal.
- Compute the kernel of $\beta := \psi \circ \varphi$.
- Consider the isomorphism of abelian groups $\iota: \mathbb{Z}^s \rightarrow \ker(\beta)$ and compute generators g_1, \dots, g_t for $\mathbb{Z}^s \cap \iota^{-1}(\mathbb{Q}_{\geq 0}^r)$.
- Define the projection $\pi: K \times K \rightarrow K, (x, y) \mapsto x$ on the first factor and return the set $\{(\pi \circ \varphi \circ \iota)(g_j); j = 1, \dots, t\}$.

Output: A set of generators for the embedded monoid $S_1 \cap S_2 \subseteq K_1 \cap K_2$.

Proof. According to Gordan's lemma, there are generators g_1, \dots, g_t for the monoid $\mathbb{Z}^s \cap \iota^{-1}(\mathbb{Q}_{\geq 0}^r)$. Let $M := \ker(\beta) \cap \mathbb{Z}_{\geq 0}^{n_1+n_2}$ and consider the diagramm

$$\begin{array}{ccccccc} \mathbb{Z}^s \cap \iota^{-1}(\mathbb{Q}_{\geq 0}^r) & \longrightarrow & M & \subseteq & \mathbb{Z}_{\geq 0}^{n_1+n_2} & \longrightarrow & S_1 \times S_2 \longrightarrow (K \times K)/\Delta \\ \text{I} \cap & & \text{I} \cap & & \text{I} \cap & & \text{I} \cap \\ \mathbb{Z}^s & \xrightarrow[\cong]{\iota} & \ker(\beta) & \subseteq & \mathbb{Z}^{n_1+n_2} & \xrightarrow{\varphi} & K \times K \xrightarrow{\psi} (K \times K)/\Delta. \\ & & & & & \searrow \beta & \nearrow \end{array}$$

With the projection $\pi: K \times K \rightarrow K, (x, y) \mapsto x$ on the first factor, we obtain

$$(\pi \circ \varphi \circ \iota)(\mathbb{Z}^s \cap \iota^{-1}(\mathbb{Q}_{\geq 0}^r)) = (\pi \circ \varphi)(M) = S_1 \cap S_2.$$

Thus, $\{(\pi \circ \varphi \circ \iota)(g_j); j = 1, \dots, t\}$ is a set of generators for $S_1 \cap S_2$. \square

Lemma 2.6. *Let $S \subseteq K$ be an embedded monoid. Consider elements $x_1, \dots, x_r \in S$ such that $\{x_1 \otimes 1, \dots, x_r \otimes 1\}$ is a set of generators for $\text{cone}(S)$. Then the finite set*

$$M := \iota^{-1} \left(\left\{ \sum_{i=1}^r \alpha_i (x_i \otimes 1); \alpha_i \in \mathbb{Q}, 0 \leq \alpha_i \leq 1 \right\} \right),$$

where ι is the map $\iota: K \rightarrow K \otimes \mathbb{Q}, w \mapsto w \otimes 1$, generates \tilde{S} as an S -module. In particular, \tilde{S} is a finitely generated S -module.

Proof. In the case of a torsion free group K , the statement on the finite generation of \tilde{S} as an S -module is Gordan's Lemma. The proof extends easily to the case of finitely generated abelian groups. \square

Algorithm 2.7 (inCondIdeal). *Input:* An embedded monoid $S \subseteq K$ and an element $w \in K$.

- Compute M as defined in Lemma 2.6.

- Use Algorithm 2.4 to test whether S contains $w + M$. Return *true* if this is the case; otherwise return *false*.

Output: *True* if w is contained in $c(\tilde{S}/S)$. Otherwise, *false* is returned.

Proof. Let $w \in K$ and consider M as defined in Lemma 2.6. According to this lemma, M generates \tilde{S} as an S -module. This means that the conductor ideal $c(\tilde{S}/S)$ contains w if and only if $w + M$ is contained in S . \square

Algorithm 2.8 (pointCondIdeal). *Input:* A spanning embedded monoid $S \subseteq K$.

- Compute $w \in K$ that defines a point in the relative interior of $\text{cone}(S)$.
- Use Algorithm 2.7 to compute the smallest integer $r \in \mathbb{Z}_{\geq 1}$ such that rw is contained in $c(\tilde{S}/S)$. Return rw .

Output: A point of the conductor ideal $c(\tilde{S}/S)$.

Proof. This Algorithm terminates since $S \subseteq K$ is spanning. \square

Here comes an example computation.

Example 2.9. We consider the embedded monoid $S \subseteq K := \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ generated by $[3, 0], [5, 0], [5, 1], [3, 2] \in K$ and perform some monoid membership and conductor ideal membership tests. Furthermore, we compute an element of the conductor ideal of $S \subseteq K$.

```
> S := matrix([[3,5,5,3],[0,0,1,2]]);
               S := [ 3  5  5  3 ]
                   [ 0  0  1  2 ]

> K := createAG(1,[3]);
               K := AG(1,[3])

> inMonoid(S,[6,2],K); inMonoid(S,[7,0],K); inMonoid(S,[8,2],K);
               true
               false
               true

> inCondIdeal(S,[6,2],K); inCondIdeal(S,[7,0],K);
  inCondIdeal(S,[8,2],K);
               false
               false
               true

> pointCondIdeal(S,K);
               [8,0]
```

We now compute generators for the intersection of the monoids $S_1, S_2 \subseteq K$ generated by $[3, 0], [5, 0], [5, 1] \in K$ and $[3, 0], [5, 0], [3, 2] \in K$, respectively.

```
> generatorsIntMonoid(S, [{1,2,3},{1,2,4}],K);
               [[3,0],[5,0],[11,1],[13,2],[15,1],[15,2]]
```

3. THE BASE POINT FREE MONOID OF MORI DREAM SPACES

We turn to Mori dream spaces and recall the necessary background from [1]. In addition, we show that the monoid of base point free Weil divisor classes of a Mori dream space X is a spanning embedded monoid in the Picard group, which will be crucial in Algorithm 4.5. We also present algorithms for computing generators of the base point free monoid and for testing whether a Weil divisor class is base point free or not.

Definition 3.1. Let D be a Weil divisor on an irreducible, normal prevariety X and consider a non-zero section $f \in \Gamma(X, \mathcal{O}_X(D))$. We call the effective divisor

$$\text{div}_D(f) := \text{div}(f) + D \in \text{WDiv}(X)$$

the D -divisor of f . The *base locus* and the *stable base locus* of the complete linear system $|D|$ or of the class $w := [D] \in \text{Cl}(X)$ are defined as

$$\text{Bs}|D| := \text{Bs}(w) := \bigcap_{f \in \Gamma(X, \mathcal{O}_X(D))} \text{Supp}(\text{div}_D(f)), \quad \mathbf{B}(w) := \bigcap_{n \in \mathbb{Z}_{\geq 0}} \text{Bs}|nD|.$$

An element $x \in \text{Bs}(w)$ is called a *base point* of w . We call $D \in \text{WDiv}(X)$ or its class $w \in \text{Cl}(X)$ *base point free*, if the base locus $\text{Bs}(w)$ is empty and *semiample*, if its stable base locus is empty. The embedded monoid $\text{BPF}(X) \subseteq \text{Pic}(X)$ of base point free Cartier divisor classes is called *base point free monoid of X* . By $\text{SAmple}(X) \subseteq \text{Cl}(X)_{\mathbb{Q}}$ and $\text{Ample}(X) \subseteq \text{Cl}(X)_{\mathbb{Q}}$, we denote the cones of semiample and ample Weil divisor classes, respectively.

Recall that a *Mori dream space* is an irreducible normal projective variety X over an algebraic closed field of characteristic zero with finitely generated divisor class group and finitely generated Cox ring

$$\text{Cox}(X) := \bigoplus_{[D] \in \text{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

Let $\mathfrak{F} := (f_1, \dots, f_r)$ be a system of pairwise non-associated $\text{Cl}(X)$ -prime generators of $\text{Cox}(X)$. Consider the homomorphism of abelian groups $Q: \mathbb{Z}^r \rightarrow \text{Cl}(X)$, $e_i \mapsto \deg(f_i)$ as well as the total coordinate space $\overline{X} := \text{Spec}(\text{Cox}(X))$. A face $\gamma_0 \preceq \gamma$ of the positive orthant $\gamma \subseteq \mathbb{Q}^r$ is called \mathfrak{F} -*face*, if there is a point $x \in \overline{X}$ with $x_i \neq 0 \Leftrightarrow e_i \in \gamma_0$ for all $1 \leq i \leq r$. The collection of *relevant faces* and the *covering collection* are

$$\text{rlv}(X) := \{\gamma_0 \preceq \gamma; \gamma_0 \text{ } \mathfrak{F}\text{-face with } \text{Ample}(X)^\circ \subseteq Q(\gamma_0)^\circ\},$$

$$\text{cov}(X) := \{\gamma_0 \in \text{rlv}(X); \gamma_0 \text{ minimal with respect to inclusion}\}.$$

Note that an ample Weil divisor class $u \in \text{Cl}(X)$ together with \mathfrak{F} and relations of $\text{Cox}(X)$ fixes a Mori dream space up to isomorphism: one can reconstruct X as GIT-quotient $p_X: X^{\text{ss}}(u) \rightarrow X$ of the set of H -semistable points $X^{\text{ss}}(u) \subseteq \overline{X}$ regarding the action of $H := \text{Spec}(\mathbb{K}(\text{Cl}(X)))$ on \overline{X} .

To any \mathfrak{F} -face $\gamma_0 \preceq \gamma$, we associate the set $X(\gamma_0) := p_X(\overline{X}(\gamma_0))$, where we have

$$\overline{X}(\gamma_0) := \{x \in \overline{X}; f_i(x) = 0 \Leftrightarrow e_i \in \gamma_0 \text{ for } 1 \leq i \leq r\} \subseteq \overline{X}.$$

With this, the Picard group $\text{Pic}(X)$ and the base locus of an element $w \in \text{Cl}(X)$ are given by

$$\text{Pic}(X) = \bigcap_{\gamma_0 \in \text{cov}(X)} Q(\text{lin}(\gamma_0) \cap E) \quad \text{and} \quad \text{Bs}(w) = \bigcup_{\substack{\gamma_0 \in \text{rlv}(X) \\ w \notin Q(\gamma_0 \cap E)}} X(\gamma_0).$$

Corollary 3.2. *The base point free monoid $\text{BPF}(Z) \subseteq \text{Pic}(Z)$ of a Mori dream space X is a spanning embedded monoid given by*

$$\text{BPF}(X) = \bigcap_{\gamma_0 \in \text{cov}(X)} Q(\gamma_0 \cap E).$$

In particular, its conductor ideal is non-empty.

Proof. The representation of $\text{BPF}(X)$ as an intersection of monoids $Q(\gamma_0 \cap E)$ is an immediate consequence of the above description of base loci. In addition, for each $\gamma_0 \in \text{cov}(X)$, the embedded monoid $Q(\gamma_0 \cap E) \subseteq Q(\text{lin}(\gamma_0) \cap E)$ is spanning. Using the above description of $\text{Pic}(X)$ together with Proposition 2.3, we obtain that $\text{BPF}(Z) \subseteq \text{Pic}(Z)$ is a spanning embedded monoid. By Proposition 2.2, this means that its conductor ideal is non-empty. \square

The following algorithms build on the maple-based software package **MDSpackage** [11]. A Mori dream space X is entered and stored in terms of an ample class u together with pairwise non-associated $\text{Cl}(X)$ -prime generators and relations of $\text{Cox}(X)$. As explained above, this data fixes a Mori dream space up to isomorphism.

Algorithm 3.3 (generatorsBPF). *Input:* A Mori dream space.

- Use **MDSpackage** to compute the covering collection of X .
- Use Algorithm 2.5 to compute generators of the intersection

$$\bigcap_{\gamma_0 \in \text{cov}(X)} Q(\gamma_0 \cap E).$$

Output: A set of generators for the embedded monoid $\text{BPF}(X) \subseteq \text{Pic}(X)$.

Algorithm 3.4 (isBasePointFree). *Input:* A Mori dream space X and a Weil divisor class $w \in \text{Cl}(X)$.

- Use Algorithm 3.3 to compute generators of $\text{BPF}(X) \subseteq \text{Pic}(X)$.
- Apply Algorithm 2.4 to w and $\text{BPF}(X)$.

Output: *True* if w is base point free. Otherwise, *false* is returned.

Using the implementation given in [8], we study the question of the existence of semiample Weil divisor classes that are not base point free. It is well known that for complete toric varieties, semiample implies base point freeness. For smooth rational projective varieties with a torus action of complexity one and Picard number two, the same statement follows immediately from the classification done in [9].

Example 3.5. We give an example of a smooth Mori dream \mathbb{K}^* -surface that admits semiample Cartier divisor classes with base points.

```
> Q := matrix([[1,-1,-1,0,0,0,0,0,0,0,0,0,0,0,0],[0,1,-1,1,0,0,0,0,0,0,0,0,0,0,0],
[0,1,0,-1,1,0,0,0,0,0,0,0,0,0,0],[0,1,0,0,-1,1,0,0,0,0,0,0,0,0,0],[0,0,0,0,0,0,
-1,1,1,0,0,0,0,0,0],[0,-1,0,0,0,1,0,-1,1,0,0,0,0,0,0],[0,0,0,1,0,0,1,0,1,1,0,0,
0,0,0],[0,1,0,0,0,0,0,1,0,1,0,0,0,0,0],[1,0,0,-1,0,0,1,0,0,0,0,1,0,0,0],[0,1,0,
0,0,0,1,0,0,0,0,1,0,0,0],[0,1,0,0,0,-1,0,0,0,0,0,0,0,1,0],[0,-1,0,0,0,1,0,0,0,0,
0,0,0,1,0]]);
```

$$Q := \begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> RL := [T[1]^5*T[2]*T[3]^4*T[4]^3*T[5]^2*T[6] + T[7]^2*T[8]*T[9]
+ T[10]^3*T[11]*T[12]^2*T[13]];
```

$$RL := [T_1^5 T_2 T_3^4 T_4^3 T_5^2 T_6 + T_7^2 T_8 T_9 + T_{10}^3 T_{11} T_{12}^2 T_{13}]$$

```
> R := createGR(RL, vars(15), [Q]);
```

$$R := \text{GR}(15, 1, [12, []])$$

```
> X := createMDS(R, relint(MDSmov(R)));
```

$$X := \text{MDS}(15, 1, 2, [12, []])$$

```
> MDSsmooth(X);
```

true

```
> COV := MDScov(X);
```

$$\begin{aligned} COV := & [\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, \{2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, \\ & \{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}, \{1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, \\ & \{1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, \{1, 2, 3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15\}, \\ & \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 15\}, \{1, 2, 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15\}, \\ & \{1, 2, 3, 4, 5, 6, 8, 10, 11, 12, 13, 14, 15\}, \{1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 15\}, \\ & \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 15\}, \\ & \{1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 14, 15\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 14, 15\}, \\ & \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 15\}, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14\}] \end{aligned}$$

```

> w := [-1,1,1,1,3,2,3,4,0,3,1,5];
                                w := [-1, 1, 1, 1, 3, 2, 3, 4, 0, 3, 1, 5]
> contains(MDSsample(X),w);
                                true
> isBasePointFree(X,w);
                                false

```

This computation shows that $w = [-1, 1, 1, 1, 3, 2, 3, 4, 0, 3, 1, 5]$ is a semiample but not base point free Cartier divisor class.

4. FUJITA BASE POINT FREE TEST

In the end of the eighties, Takao Fujita conjectured the following:

Conjecture 4.1 (Fujita's base point free conjecture [10]). *Let X be a n -dimensional smooth projective variety with canonical class K_X and let \mathcal{L} be an ample Cartier divisor class. Then the following holds:*

$K_X + m\mathcal{L}$ is base point free for all $m \geq n + 1$.

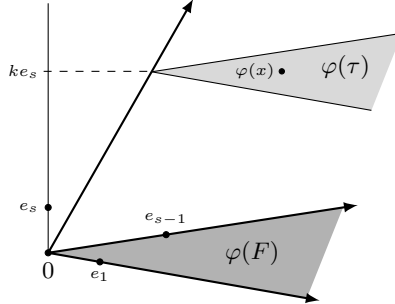
In order to test whether an arbitrary Mori dream space X with known canonical class fulfills Fujita's base point free conjecture, we need to test whether $K_X + m\mathcal{L}$ is an element of $\text{BPF}(X)$ for all $m \geq \dim(X) + 1$ and for all ample Cartier divisor classes \mathcal{L} . Since we can only carry out finitely many tests, we encounter two problems: firstly, we need to bound m and secondly, we need to find a finite validation set of Cartier divisor classes \mathcal{L} . In this section, we introduce our solution to these problems and also present some examples of applying our test algorithm.

Remark 4.2. Algorithm 4.5 applies to Mori dream spaces with known canonical class. For instance, if $\text{Cox}(X)$ is a complete intersection, there is a concrete formula for the canonical class in terms of generators and relations of $\text{Cox}(X)$ [1]. Note that all irreducible normal rational projective varieties with a torus action of complexity one have a complete intersection Cox ring. In addition, there are formulas for the canonical class of spherical varieties, see [3, 14].

Construction 4.3. Let K^0 be a lattice. Consider an s -dimensional cone $\sigma \subseteq K_{\mathbb{Q}}^0$ with some facet $F \preceq \sigma$ and let $x \in K^0$. The x -facet parallel of F is the convex set

$$\tau := ((x \otimes 1) + \text{lin}_{\mathbb{Q}}(F)) \cap \sigma \subseteq K_{\mathbb{Q}}^0.$$

Let $\varphi: K^0 \rightarrow \mathbb{Z}^n$ be an isomorphism of \mathbb{Z} -modules such that $\varphi(\sigma) \subseteq \text{cone}(e_1, \dots, e_s)$ and $\varphi(F) \subseteq \text{lin}_{\mathbb{Q}}(e_1, \dots, e_{s-1})$ holds, where e_1, \dots, e_n denote the canonical base vectors of the rational vector space \mathbb{Q}^n . Consider $\varphi(x) \in \mathbb{Z}^n$ and its s -th coordinate $k := \varphi(x)_s \in \mathbb{Z}$ as indicated in the figure below. With this, we call τ the k -th facet parallel of F .



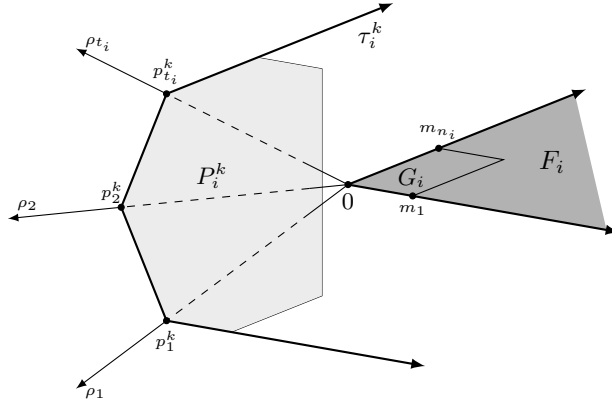
Setting 4.4. Let X be a Gorenstein Mori dream space and consider the base point free monoid $S := \text{BPF}(X) \subseteq K := \text{Pic}(X)$. We denote by F_1, \dots, F_r the facets of $\sigma := \text{cone}(w^0 \otimes 1; w \in S) \subseteq K_{\mathbb{Q}}^0$. Let $1 \leq i \leq r$ and let $m_1, \dots, m_{n_i} \in S$ be those elements such that m_j^0 is minimal with the property that $m_j^0 \otimes 1$ is contained in a ray of F_i . Consider the polytope

$$G_i := \left\{ \sum_{j=1}^{n_i} a_j (m_j^0 \otimes 1); a_j \in \mathbb{Q}, 0 \leq a_j \leq 1 \right\} \subseteq F_i$$

as indicated in the figure below and let $\rho_1, \dots, \rho_{t_i}$ be the rays of σ that are not contained in F_i . We denote by τ_i^k the k -th facet parallel of F_i . For each facet parallel τ_i^k , we denote by $p_j^k \in K_{\mathbb{Q}}$, $1 \leq j \leq t_i$, the point that is the intersection of ρ_j and τ_i^k . With the canonical embedding $\iota_0: K^0 \rightarrow K_{\mathbb{Q}}^0$, $w \mapsto w \otimes 1$, we define

$$\begin{aligned} P_i^k &:= (\text{conv}(p_1^k, \dots, p_{t_i}^k) + G_i) \cap \sigma^\circ \subseteq \tau_i^k \quad \text{and} \\ Gp_i^k &:= \iota_0^{-1}(P_i^k) \times K^{\text{tor}} \subseteq K, \end{aligned}$$

where σ° denotes the relative interior of σ . Consider the canonical class $\mathcal{K}_X \in K$ of X and let $C \in c(\tilde{S}/S)$. For $1 \leq i \leq r$ let $\alpha_i \in \mathbb{Z}$ such that $(-\mathcal{K}_X^0 + C^0) \otimes 1 \in \tau_i^{\alpha_i}$ holds and set $\nu := \max(\alpha_i; 1 \leq i \leq r)$.



The above mentioned problems, namely bounding m and finding a finite validation set of Cartier divisor classes, are tackled by computing a point of the conductor ideal of $\text{BPF}(X)$ and by only considering the Cartier divisor classes defining a point in the polytopes P_i^k of the first few facet parallels τ_i^k of each facet $F_i \preceq \sigma$.

Algorithm 4.5 (fujitaBpf). *Input:* A Mori dream space X and its canonical class \mathcal{K}_X .

- If X is not Gorenstein return *false*.
- Use Algorithm 2.5 to compute generators of $S := \text{BPF}(X)$.
- Use Algorithm 2.8 to compute a point $C \in c(\tilde{S}/S)$.
- Compute the facets F_1, \dots, F_r of $\text{cone}(S)$ and $\alpha_1, \dots, \alpha_r$ as well as ν as defined in Setting 4.4.
- For each $1 \leq i \leq r$ do
 - for each $\dim(X) + 1 \leq m \leq \nu - 1$ do
 - * for each $1 \leq k \leq \lfloor \frac{\alpha_i - 1}{m} \rfloor$ use Algorithm 2.4 to test whether $\mathcal{K}_X + m Gp_i^k \subseteq S$ holds.
- Return *false* if there is $1 \leq i \leq r$, $\dim(X) + 1 \leq m \leq \nu - 1$, $1 \leq k \leq \lfloor \frac{\alpha_i - 1}{m} \rfloor$, and $\mathcal{L} \in Gp_i^k$ such that $\mathcal{K}_X + m\mathcal{L}$ is not contained in S . Otherwise, return *true*.

Output: *True* if X fulfills Fujita's base point free conjecture, i.e. if $\mathcal{K}_X + m\mathcal{L}$ is base point free for all $m \geq \dim(X) + 1$ and all ample Cartier divisor classes \mathcal{L} . Otherwise, *false* is returned.

Before presenting a proof of Algorithm 4.5, we first give two examples of applying it to Mori dream spaces.

Example 4.6. Here we give an example of a six-dimensional smooth Mori dream space that does fulfill Fujita's base point free conjecture.

```
> Q := matrix([[1,1,2,0,1,1,1,-1,0,0],[0,0,-1,1,0,-1,-1,1,0,0],[0,36,36,0,18,49,49,-48,1,1]]);
```

$$Q = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & 36 & 36 & 0 & 18 & 49 & 49 & -48 & 1 & 1 \end{bmatrix}$$

```
> RL := [T[1]*T[2]+T[3]*T[4]+T[5]^2];
```

$$RL := [T_1 T_2 + T_3 T_4 + T_5^2]$$

```
> R := createGR(RL,vars(10),[Q]);
```

$$R := GR(10, 1, [3, []])$$

```
> X := createMDS(R,createMDS(R,[1,1,50]));
```

$$X := MDS(10, 1, 6, [3, []])$$

```
> MDSsmooth(X);
```

true

Since $\text{Cox}(X)$ is a complete intersection, we may use the formula presented in [1] to compute the canonical class of X : we obtain $\mathcal{K}_X = [-4, 1, -106] \in \mathbb{Z}^3$.

```
> fujitaBPF(X,[-4,1,-106]);
```

true

Example 4.7. Here we give an example of a locally factorial Mori dream space that does not fulfill Fujita's base point free conjecture.

```
> Q := matrix([[0,0,1,0,0,1,1,0,1],[1,1,0,1,1,0,1,1,2]]);
```

$$Q = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 2 \end{bmatrix}$$

```
> RL := [T[1]*T[2]^7+T[3]^8 +T[4]*T[5]^7*T[6]^8+T[7]^8];
```

$$RL := [T_1 T_2^7 T_3^8 + T_4 T_5^7 T_6^8 + T_7^8]$$

```
> R := createGR(RL,vars(9),[Q]);
```

$$R := GR(9, 1, [2, []])$$

```
> X := createMDS(R,[1,3]);
```

$$X := MDS(9, 1, 6, [2, []])$$

```
> MDSisfact(X);
```

true

```
> MDSisquasismooth(X);
```

false

Since $\text{Cox}(X)$ is a complete intersection, we may use the formula presented in [1] to compute the canonical class of X : we obtain $\mathcal{K}_X = [4, 0] \in \mathbb{Z}^2$.

```
> fujitaBPF(X,[4,0]);
```

false

We now turn to the proof of Algorithm 4.5.

Lemma 4.8. *In the setting of 4.4, the following are equivalent:*

- (i) $\mathcal{K}_X + m\mathcal{L} \in S$ holds for all $m \geq \dim(X) + 1$ and for all ample Cartier divisor classes \mathcal{L} , i.e. X fulfills Fujita's base point free conjecture.
- (ii) $\mathcal{K}_X + m\mathcal{L} \in S$ holds for all $\nu - 1 \geq m \geq \dim(X) + 1$ and for all ample Cartier divisor classes \mathcal{L} .

Proof. Only implication “(ii) \Rightarrow (i)” needs to be proven. Let $m \geq \dim(X) + 1$. If $m \leq \nu - 1$ holds, then $\mathcal{K}_X + m\mathcal{L} \in S$ follows by (ii). If $m > \nu - 1$ holds, we obtain

$$m\mathcal{L} \otimes 1 \in ((-\mathcal{K}_X + C) \otimes 1) + \text{cone}(S).$$

Since $C \in c(\tilde{S}/S)$ holds, we conclude $\mathcal{K}_X + m\mathcal{L} \in S$. □

Lemma 4.9. *In the setting of 4.4, the following are equivalent for $m \in \{\dim(X) + 1, \dots, \nu - 1\}$:*

- (i) $\mathcal{K}_X + m\mathcal{L} \in S$ holds for all ample Cartier divisor classes \mathcal{L} .
- (ii) For all $1 \leq i \leq r$ and for all $1 \leq k \leq \lfloor \frac{\alpha_i - 1}{m} \rfloor$, we have $\mathcal{K}_X + m\mathcal{L} \in S$ for all $\mathcal{L} \in \iota_0^{-1}(\tau_i^k \cap \sigma^\circ) \times K^{\text{tor}}$.

Proof. Only implication “(ii) \Rightarrow (i)” needs to be proven. Consider an ample Cartier divisor class \mathcal{L} , i.e.

$$\mathcal{L} \in \iota_0^{-1}(\sigma^\circ) \times K^{\text{tor}}$$

holds. Let $\beta_1, \dots, \beta_r \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{L}^0 \otimes 1 \in \tau_i^{\beta_i}$ holds. If $\beta_i \leq \lfloor \frac{\alpha_i - 1}{m} \rfloor$ holds for some $1 \leq i \leq r$, then $\mathcal{K}_X + m\mathcal{L} \in S$ follows by (ii). Now assume that $\beta_i > \lfloor \frac{\alpha_i - 1}{m} \rfloor$ holds for all $1 \leq i \leq r$. We obtain $m\beta_i \geq \alpha_i$ for all $1 \leq i \leq r$, which shows

$$m\mathcal{L} \otimes 1 \in ((-\mathcal{K}_X + C) \otimes 1) + \text{cone}(S).$$

Since $C \in c(\tilde{S}/S)$ holds, we conclude $\mathcal{K}_X + m\mathcal{L} \in S$. \square

Lemma 4.10. *In the setting of 4.4, let $1 \leq i \leq r$, $1 \leq k \leq \lfloor \frac{\alpha_i - 1}{m} \rfloor$ and consider an ample Cartier divisor class $\mathcal{L} \in \iota_0^{-1}(\tau_i^k \cap \sigma^\circ) \times K^{\text{tor}}$. Then there are $y \in Gp_i^k$ and $\alpha_j \in \mathbb{Z}_{\geq 0}$ such that we have*

$$\mathcal{L} = y + \sum_{j=1}^{n_i} \alpha_j m_j.$$

Proof. Observe that $\tau_i^k = \text{conv}(p_1^k, \dots, p_{t_i}^k) + \text{cone}(G_i)$ holds. Hence there are rational numbers $a_j, b_\ell \in \mathbb{Q}_{\geq 0}$, $\sum_{j=1}^{t_i} a_j = 1$, such that

$$\mathcal{L} = \left(\sum_{j=1}^{t_i} a_j p_j^k + \sum_{\ell=1}^{n_i} b_\ell m_\ell^0, \mathcal{L}^{\text{tor}} \right)$$

holds. We obtain $\mathcal{L} = y + \sum_{\ell=1}^{n_i} \lfloor b_\ell \rfloor m_\ell$, where y is given as

$$y := \left(\sum_{j=1}^{t_i} a_j p_j^k, \mathcal{L}^{\text{tor}} - \sum_{\ell=1}^{n_i} b_\ell m_\ell^{\text{tor}} \right) + \sum_{\ell=1}^{n_i} (b_\ell - \lfloor b_\ell \rfloor) m_\ell.$$

If $y^0 \otimes 1 \in \sigma^\circ$ holds, we achieved the required representation of \mathcal{L} . Now assume that $y^0 \otimes 1$ is not contained in σ° . Since $\mathcal{L}^0 \otimes 1$ is contained in σ° , there is $1 \leq \ell \leq n_i$ with $\lfloor b_\ell \rfloor \neq 0$. Without loss of generality we assume that $\lfloor b_1 \rfloor, \dots, \lfloor b_{\ell_0} \rfloor > 0$ and $\lfloor b_{\ell_0+1} \rfloor = \dots = \lfloor b_{n_i} \rfloor = 0$ hold for some $1 \leq \ell_0 \leq n_i$. Then the required representation is given by $\mathcal{L} = y' + \sum_{1 \leq \ell \leq \ell_0} (\lfloor b_\ell \rfloor - 1) m_\ell$, where we have

$$y' := y + \sum_{\ell=1}^{\ell_0} m_j.$$

\square

Lemma 4.11. *In the setting of 4.4, let $\dim(X) + 1 \leq m \leq \nu - 1$, $1 \leq i \leq r$ and $1 \leq k \leq \lfloor \frac{\alpha_i - 1}{m} \rfloor$. Then the following are equivalent:*

- (i) $\mathcal{K}_X + m\mathcal{L} \in S$ holds for all $\mathcal{L} \in \iota_0^{-1}(\tau_i^k \cap \sigma^\circ) \times K^{\text{tor}}$.
- (ii) $\mathcal{K}_X + m\mathcal{L} \in S$ holds for all $\mathcal{L} \in Gp_i^k$.

Proof. Since $Gp_i^k \subseteq \iota_0^{-1}(\tau_i^k \cap \sigma^\circ) \times K^{\text{tor}}$ holds, only implication “(ii) \Rightarrow (i)” needs to be proven. Note that this is an immediate consequence of Lemma 4.10. \square

Proof of Algorithm 4.5. We need to show that X fulfills Fujita’s base point free conjecture if and only if the above algorithm returns *true*. This can be seen as follows: if X is not Gorenstein, then $\mathcal{K}_X + m\mathcal{L}$ is not a Cartier divisor class; in particular, it is not base point free. Now assume that X is Gorenstein. According

to Corollary 3.2, the embedded monoid $\text{BPF}(X) \subseteq \text{Pic}(X)$ is spanning, i.e. we can apply Algorithm 2.8 and compute a point of its conductor ideal. Lemma 4.8 shows that we can bound m by $\nu - 1$; Lemmata 4.9 and 4.11 prove that the sets Gp_i^k , $1 \leq i \leq r$, $1 \leq k \leq \lfloor \frac{\alpha_i - 1}{m} \rfloor$, serve as validation sets of Cartier divisor classes. \square

REFERENCES

- [1] I. Arzhantsev, U. Derenthal, J. Hausen, A. Laface: *Cox rings*. Cambridge Studies in Advanced Mathematics no. 144, Cambridge Univ. Press, Cambridge (2014).
- [2] C. Birkar, P. Cascini, C. Hacon, J. McKernan: *Existence of minimal models for varieties of log general type*. J. Amer. Math. Soc. 23 (2010), 405–468.
- [3] M. Brion: *Curves and divisors in spherical varieties*. In Algebraic groups and Lie groups, Austral. Math. Soc. Lect. Ser., vol. 9, Cambridge Univ. Press, Cambridge (1997), 21–34.
- [4] M. Brion, F. Knop: *Contractions and flips for varieties with group action of small complexity*. J. Math. Sci. Univ. Tokyo vol. 1 (1994), no. 3, 641–655.
- [5] W. Bruns, J. Gubeladze: *Polytopes, Rings, and K-Theory*. Springer, Heidelberg (2009).
- [6] L. Ein, R. Lazarsfeld: *Global generation of pluricanonical and adjoint linear series on smooth projective threefolds*. J. Amer. Math. Soc. vol. 6 no. 4 (1993), 875–903.
- [7] M. Franz: *Convex a Maple package for convex geometry*. Available at <http://www.math.uwo.ca/~mfranz/convex/>.
- [8] A. Fahrner: *Monoid Package - a package for monoids and base point free problems of Mori dream spaces*. Available at <http://www.math.uni-tuebingen.de/user/fahrner/MonoidPackage/>.
- [9] A. Fahrner, J. Hausen, M. Nicolussi: *Smooth projective varieties with a torus action of complexity 1 and Picard number 2*. To appear in Annali della Scuola Normale Superiore di Pisa.
- [10] T. Fujita: *Problems*. In: Birational Geometry of Algebraic Varieties: Open Problems, Katata Symposium of the Taniguchi Foundation (1988), 42–47.
- [11] J. Hausen, S. Keicher: *A software package for Mori dream spaces*. LMS J. Comput. Math. vol. 18 no. 1 (2015), 647–659.
- [12] Y. Hu, S. Keel: *Mori dream spaces and GIT*. Dedicated to William Fulton on the occasion of his 60th birthday. Michigan Math. J. 48 (2000), 331–348.
- [13] Y. Kawamata: *On Fujita’s freeness conjecture for 3-folds and 4-folds*. Math. Ann. vol. 308 no. 3 (1997), 491–505.
- [14] D. Luna: *Grosses cellules pour les variétés sphériques*. In Algebraic Groups and Lie Groups, ed. by G. I. Lehrer, Australian Math. Soc. Lecture, Series 9 (1997), 267–280.
- [15] J. McKernan: *Mori dream spaces*. Jpn. J. Math. 5 (2010), no. 1, 127–151.
- [16] I. Reider: *Vector bundles of rank 2 and linear systems on algebraic surfaces*. Ann. of Math. 2 (1988), vol. 127 no.2, 309–316.
- [17] F. Ye, Z. Zhu: *On Fujita’s freeness conjecture in dimension 5*. ArXiv e-prints 2015.

MATHEMATISCHES INSTITUT, UNIVERSITÄT TÜBINGEN, AUF DER MORGENSTELLE 10, 72076 TÜBINGEN, GERMANY

E-mail address: fahrner@math.uni-tuebingen.de